UiO Tropical Geometry Learning Seminar

Session 1:

Tropical Algebra, Curves & Hypersurfaces

Tuesday 31. August 2021

Overview:

- Tropical algebra:
  - Tropical semifield
  - Tropical polynomials
- Tropical curves:
  - Definition in $\mathbb{R}^2$
  - Dual subdivision & how to draw tropical curves
  - Balancing condition.
- Tropical hypersurfaces

The semifield $\mathbb{T}$

$$\mathbb{T} := \mathbb{R} \cup \{-\infty\}$$

$x, y \in \mathbb{T}$

$$x \oplus y := \max\{x, y\}$$

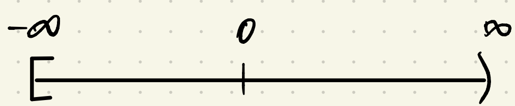
$$x \odot y := x + y$$

Semiring: no additive inverse

$x^{\odot -1} =$ $\rightsquigarrow$ Semifield

$x^{\odot -1} \odot x =$

$\mathbb{T}$:



$2 \oplus 3 = \qquad = 3 \oplus 2$

$2 \oplus 2 =$

$x \oplus x =$

$x \oplus -\infty =$

$x \oplus y = -\infty \longrightarrow$

$1 \odot 2 =$

$0 \odot x =$

$x^{\odot 2} =$

$$\begin{aligned} x \odot (y \oplus z) &= x + \max(y, z) \\ &= \max(x+y, x+z) \\ &= (x \odot y) \oplus (x \odot z) \end{aligned}$$

Tropical polynomials

$p : \mathbb{T} \to \mathbb{T}$

$$P(x) = \bigoplus_{i=0}^{d} a_i \odot x^{\odot i}$$

Deg 0: $P(x) = a_0$

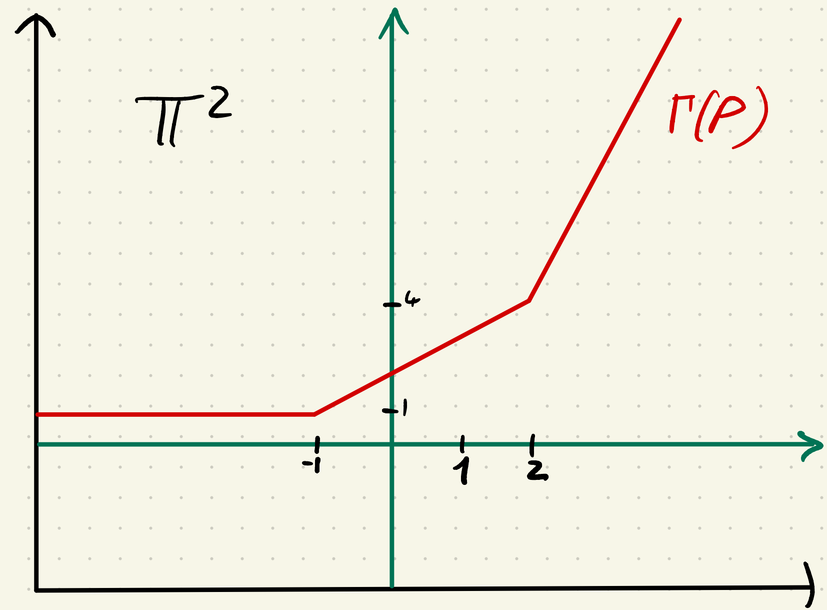
Deg 1: $P(x) = a_0 \oplus a_1 \odot x$
$= \max(a_0, a_1 + x)$

Deg 2: $P(x) = a_0 \oplus a_1 \odot x \oplus a_2 \odot x^{\odot 2}$
$= \max(a_0, a_1 + x, a_2 + 2x)$

$\vdots$

$P(x) = 1 \oplus 2 \odot x \oplus x^{\odot 2}$ $\qquad a_2 = 0$
$= \max\{1, x+2, 2x\}$

$$\Gamma(P) = \{(x, P(x))\} \subseteq \mathbb{T}^2$$



Roots of tropical polynomials

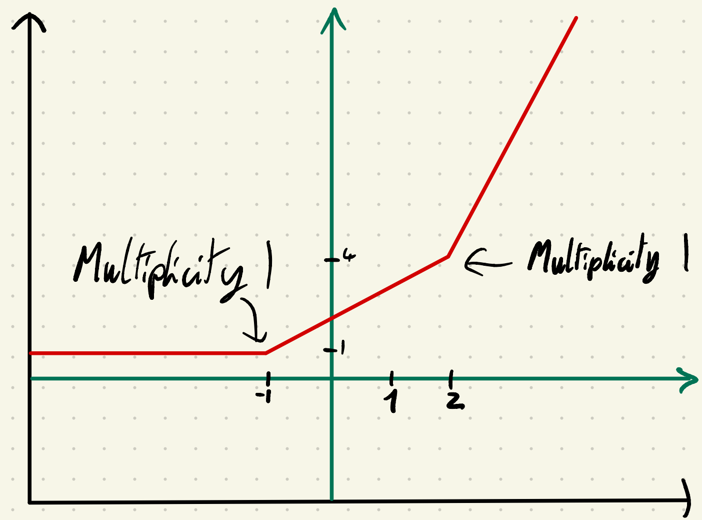
Def: $x_0 \in \mathbb{T}$ is a root of P if the value $P(x_0)$ is attained at least twice.
$(\exists i \neq j \text{ s.t. } a_i \odot x^i = a_j \odot x^j)$

$\Leftrightarrow$ Corners in $\Gamma(P)$.

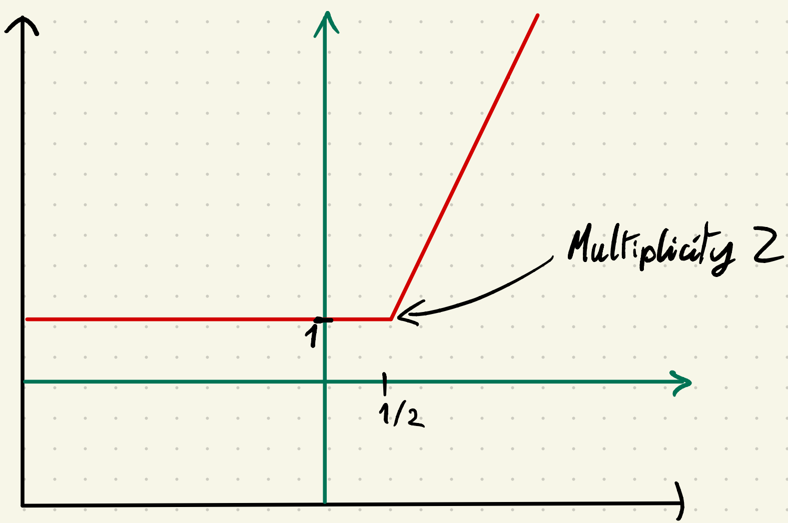
The multiplicity is the maximum of $|i-j|$ for all i,j realising $P(x_0)$

$\Leftrightarrow$ Slope differences

$$P(x) = 1 \oplus 2 \odot x \oplus x^{\odot 2} \qquad a_2 = 0$$
$$= \max\{1, x+2, 2x\}.$$



$$P(x) = 1 \oplus x \oplus x^{\odot 2}$$
$$= \max\{1, x, 2x\}$$



Proposition

A degree d tropical polynomial has exactly d roots, counted with multiplicity.

$\rightsquigarrow$ Tropical semifield is algebraically closed.

Proof: Exercise

(hint: compare steepness of slopes)

Tropical curves

$$P(x,y) = \bigoplus_{i,j} a_{ij} \odot x^{\odot i} \odot y^{\odot j}$$

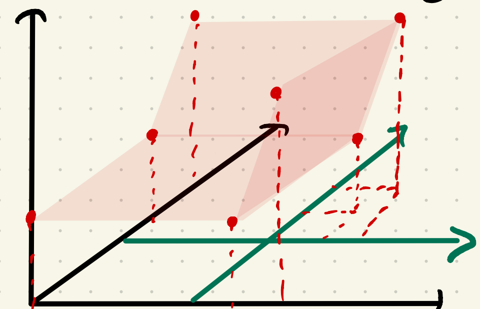
Def: The tropical curve C of P is

$$\left\{ (x_0, y_0) \in \mathbb{T}^2 \;\middle|\; \begin{array}{l} \exists\, (i,j) \neq (k,l) \\ \text{s.t. } a_{ij} x_0^i y_0^j = a_{kl} x_0^k y_0^l \end{array} \right\}$$

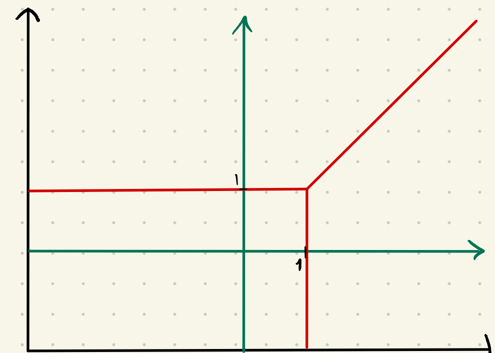
i.e. points where the max is attained twice.

(Set of points where $\Gamma(P) \subset \mathbb{T}^3$ is nonlinear)

$$P(x,y) = 1 \oplus x \oplus y$$
$$= \max\{1, x, y\}$$



- $1 = x \geq y \rightsquigarrow \{(1,y) \mid y \leq 1\}$
- $1 = y \geq x \rightsquigarrow \{(x,1) \mid x \leq 1\}$
- $x = y \geq 1 \rightsquigarrow \{(t,t) \mid 1 \leq t\}$



Dual subdivision & how to draw tropical curves in $\mathbb{R}^2$

$$P(x,y) = \bigoplus_{i,j} a_{ij} x^i y^j$$

$$\Delta(P) = \mathrm{Conv}\{(i,j) \in \mathbb{Z}^2 \mid a_{ij} \neq -\infty\}$$

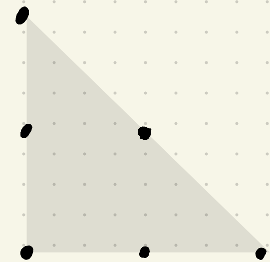
$$\mathrm{Height}: \Delta(P) \longrightarrow \mathbb{R}$$
$$(i,j) \longmapsto a_{ij}$$

$$\Gamma(\mathrm{Height}) \subset \mathbb{R}^3$$

$\rightsquigarrow$ Shadow leads to a subdivision of $\Delta(P)$

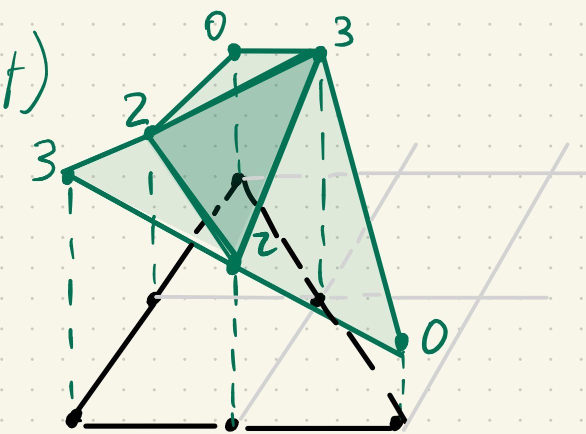
$$P(x,y) = \underset{(0,0)}{3} \oplus \underset{(1,0)}{2x} \oplus \underset{(0,1)}{2y} \oplus \underset{(1,1)}{3xy} \oplus \underset{(0,2)}{y^2} \oplus \underset{(2,0)}{x^2}$$

$\Delta(P)$



$\rightsquigarrow$

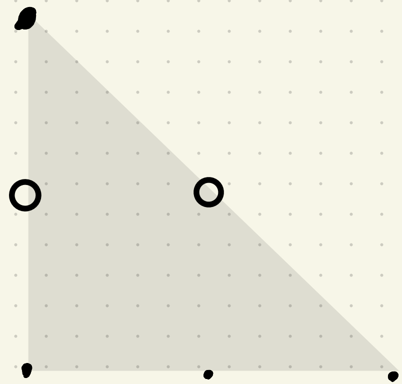
$\Gamma(\mathrm{Height})$



Example & weights

$$P(x,y) = 0 \oplus x \oplus y^2 \oplus (-1)x^2$$

$\Delta(P)$:

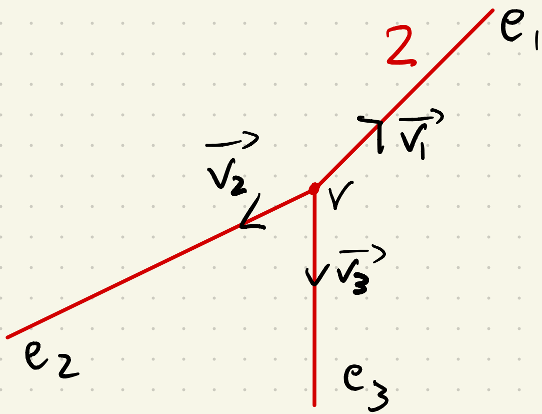


The **weight** of an edge e of C is

$$w_e = \mathrm{Card}\left(\Delta_e \cap \mathbb{Z}^2\right) - 1$$

A **tropical curve** in $\mathbb{R}^2$ is a weighted graph with unbounded edges.

Balancing condition



$\vec{v_1} = (1,1)$

$\vec{v_2} = (-2,-1)$

$\vec{v_3} = (0,-1)$

Proposition: v vertex in tropical curve C, edges $e_1, \dots, e_k$, with weights $w_1, \dots, w_k$, and primitive integer vectors $\vec{v_1}, \dots, \vec{v_k}$

Then

$$\sum_{i=1}^{k} w_i \vec{v_i} = 0$$

$\rightarrow$ Tropical curves are balanced

Theorem (Mikhalkin 04)

Any balanced graph in $\mathbb{R}^2$ is a tropical curve.

Tropical hypersurfaces in $\mathbb{R}^n$

$$P(x_1,\dots,x_n)=\bigoplus_{\vec{i}\in A} a_{\vec{i}}\,\vec{x}^{\,\vec{i}} = \max_{\vec{i}\in A}\{a_{\vec{i}}+\langle \vec{i},\vec{x}\rangle\}$$

$A\subset(\mathbb{Z}_{\geq 0})^n$ finite, $\vec{x}^{\,\vec{i}}=x_1^{\odot i_1}\odot x_2^{\odot i_2}\cdots\odot x_n^{\odot i_n}$

$V(P)=\{$points in $\mathbb{R}^n$ where max is attained twice$\}$

$\iff$

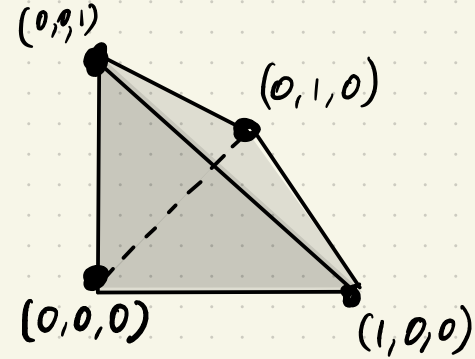
Dual to the subdivision of

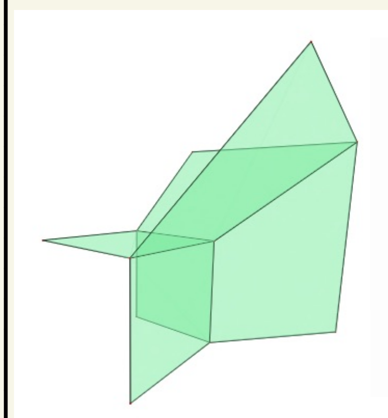
$$\Delta(P)=\mathrm{Conv}\left(\left\{\{i\in(\mathbb{Z}_{\geq 0})^n \,/\, a_i\neq-\infty\}\right\}\right)$$

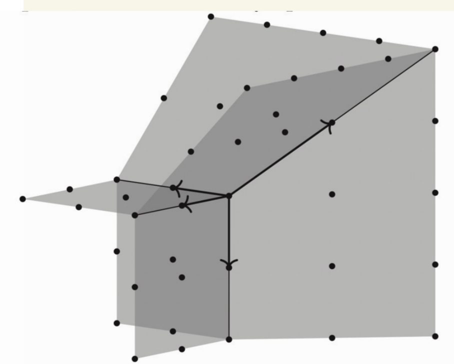
induced by coeffs of P.

$$P(x,y,z)=x\oplus y\oplus z\oplus 0$$

$\Delta(P)$:







Exercises

(1) Prove that the tropical semifield is algebraically closed.

(2) Prove that x_0 is a root of order k of a tropical polynomial $P(x)$ iff $P(x) = (x \oplus x_0)^{\odot k} \odot Q(x)$ for some tropical polynomial $Q(x)$ of which x_0 is not a root.

(3) Let $a \in \mathbb{R}$ and $b, c \in \mathbb{T}$. Determine the roots of $b \oplus a \odot x$ and $c \oplus b \odot x \oplus a \odot x^{\odot 2}$

(4) Draw the tropical curves defined by $P(x,y) = 5 \oplus 5x \oplus 5y \oplus 4xy \oplus 1y^2 \oplus x^2$ and $Q(x,y) = 7 \oplus 4x \oplus y \oplus 4xy \oplus 3y^2 \oplus (-3)x^2$, and their dual subdivisions.

(5) Show that a tropical curve of degree d has at most d^2 vertices